COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 8 (4) 1988 373—377

EQUIVALENT FACTOR MATROIDS OF GRAPHS

D. K. WAGNER

Received May 6, 1986
Revised August 28, 1987

The factor matroid of a graph G is the matric matroid of the vertex-edge incidence matrix of G interpreted over the real numbers. This paper presents a constructive characterization of the graphs hat have the same factor matroid as a given 4-connected bipartite graph.

1. Introduction

Let G be a finite, undirected, loopless graph, and let N be the vertex-edge incidence matrix of G. Denote by $M_F(N)$ the matric matroid of N with linear independence interpreted over a field F. (Matric matroids are also known as representable matroids — see Welsh [5].) When $F=GF(2)$, $M_F(N)$ is the well-known polygon matroid of G. When $F=\mathbf{R}$, $M_F(N)$ is the not-so-well-known factor matroid of G.

This paper addresses the following question: For a given graph G, what other graphs have the same factor matroid as G? An obvious second question is: What other graphs have the same polygon matroid as G? This second question has been answered nicely by Whitney [6]. Whitney's answer to the second question also provides a partial answer to the first since if G is bipartite, then the factor matroid and the polygon matroid are the same. Thus, Whitney has given a complete answer to the question: For a given bipartite graph G, what other bipartite graphs have the same factor matroid as G?

More precisely, this paper addresses the question: For a given bipartite graph G, what other non-bipartite graphs have the same factor matroid as G? The main result of this paper is an answer to this question when G is 4-connected.

2. Preliminaries

A basic knowledge of graph and matroid theory is assumed. Most of the terminology is consistent with Bondy and Murty [1] and Welsh [5]. All graphs considered are finite and undirected with loops and parallel edges allowed.

Let $G=(V, E)$ be a graph. Where $G[C]$ denotes the subgraph induced by C, the set $\{G\subseteq E|G[C]$ is a cycle$\}$ is the collection of circuits of a matroid on E, called the *polygon* (or *cycle*) matroid of G and denoted by $P(G)$.

Research partially supported by NSF Grant ESS-8307796 and Office of Naval Research Grant N00014-86-K-0689.

AMS subject classification code (1980): 05 B 35.

A cycle is *odd* (respectively, *even*) if it has an odd (respectively, even) number of edges. A *bicycle* is a graph that is a subdivision of one of the graphs in Figure 1. A bicycle is *odd* if both of its cycles are odd.

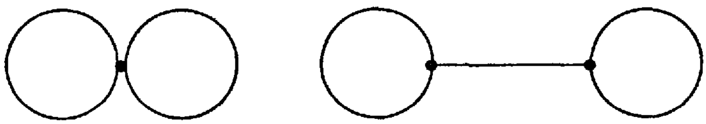

Fig. 1

The set $\{C \subseteq E | G[C]$ is an even cycle or odd bicycle$\}$ is known to be the collection of circuits of a matroid on E, called the *factor* matroid of G and denoted $F(G)$; see Doob [2], Simões-Pereira [3], Tutte [4] and Zaslavsky [7]. (The name factor matroid is new and is used because of the relationship between this matroid and factors in graphs — see Tutte [4]. Other names include even-circle matroid and unoriented-cycle matroid.) A cocircuit of $F(G)$ is a minimal set of edges D such that $G \setminus D$ has more bipartite components than G, where an isolated vertex is considered as a bipartite component. The matroid rank of a set A of edges is given by the number of vertices in $G[A]$ minus the number of bipartite components of $G[A]$. For details on these and other graph-theoretic features of $F(G)$, see Zaslavsky [7].

If no edge of G is a loop, then the definitions of the polygon and factor matroid of G ccincide with those given in the Introduction. If G has a loop, then one must be careful in the definition of the incidence matrix. More precisely, define N to be the matrix whose (i, j) entry is given by n_{ij}, where n_{ij} is defined to be 1 if vertex v_i is incident to edge e_j, and 0 otherwise. Define a second matrix N' exactly as N with the exception that if edge e_j is a loop, then n'_{ij} is 0 for all i. Then $F(G)$ is the matric matroid of N (over $\mathbf{R}$), and $P(G)$ is the matric matroid of N' (over $GF(2)$). Call N the *real* incidence matrix of G, and N' the *binary* incidence matrix of G. Note if G is bipartite, then these two matrices coincide.

3. 4-Connectivity

This section contains the main result of the paper — a characterization of the class of graphs having the same factor matroid as a given 4-connected bipartite graph. The next two results describe "factor-matroid-preserving" operations. Throughout this section $G=(V, E)$ denotes a bipartite graph.

Let v be a vertex of G and let $e_1, \ldots, e_k$ be the edges of G incident to v. The other end of e_i is denoted by v_i, for $1 \leq i \leq k$. Define G' to be the graph obtained from G by first deleting v, and then adding the edges $e_1, \ldots, e_k$ as loops incident to $v_1, \ldots, v_k$, respectively. The graph G' is said to be obtained from G by a *rolling* at v. Observe that the only odd cycles of G' are the loops $e_1, \ldots, e_k$.

Theorem 3.1. *If G' is obtained from G by a rolling at v, then $F(G)=F(G')$.*

Proof. The proof that follows, as well as the proof of (3.2), were pointed out by one of the referees. Alternatively, one could check that the set of cycles of G corresponds precisely to the set of even cycles and odd bicycles of G'.

Let N be a real incidence matrix of G. Since G is bipartite, deleting any row of N does not affect linear dependence among the columns. The result now follows by noting that deleting the row corresponding to vertex v yields a real incidence matrix of G'. ▮

Let u and v be vertices of G that are on opposite sides of the vertex partition of G. Define G' to be the graph obtained from G by identifying u and v. Thus, an edge joining u and v in G is a loop of G'. This operation is termed an *identification* of u and v.

Theorem 3.2. *If G' is obtained from G by an identification of u and v, then $F(G)=F(G')$.*

Proof. Let N be a real incidence matrix of G, and let r_u and r_v be the rows corresponding to u and v, respectively. Observe that a real incidence matrix for G' can be obtained by first replacing r_u and r_v by their sum and then by scaling the columns of edges incident to both u and v in G so that all nonzero entries are 1. Neither of these operations affect linear dependence among the columns and so the result follows. ▮

The main result is that when G is 4-connected, then every graph having the same factor matroid as G is obtainable from G by either a rolling or an identification. Before proving this result, some known preliminary results are stated.

Let $G=(V, E)$ be a connected graph and suppose $\{E_1, E_2\}$ is a partition of E. Then $\{E_1, E_2\}$ is a *k-separation* of G if $|V(G[E_1])|>k<|V(G[E_2])|$ and $|V(G(E_1)])\cap \cap V(G[E_2])|=k$. The graph G is *n-connected,* for a positive integer n, if it has no k-separation for $k<n$.

Two matroids on the same element set are defined to be equal if they have precisely the same circuits. Two graphs on the same edge set are defined to be equal if there exists a bijection between their vertex sets that preserves vertex-edge incidence.

Whitney [6] characterized the class of graphs having the same polygon matroid as a given graph. Restricted to 3-connected graphs, Whitney's result is the following.

Theorem 3.3. *Let H be a loopless 3-connected graph and let H' be a connected graph. If $P(H)=P(H')$, then $H=H'$.* ▮

Let $u\in V$ and $T\subseteqq V-\{u\}$ with $T=\{u_1, \ldots, u_t\}$. Then $F\subseteqq E$ is a *(u, T)-fan* if $G[F]$ is a tree such that the degree of u in $G[F]$ is t, the degree of u_i in $G[F]$ is 1, for $1\leqq i\leqq t$, and the degree of all other vertices of $G[F]$ is 2. The next result follows easily from Menger's theorem.

Theorem 3.4. *Let $G=(V, E)$ be a n-connected graph with at least $n+1$ vertices. Let $u\in V$ and $T\subseteqq V-\{u\}$ such that $|T|=n$. Then there exists a (u, T)-fan of G.* ▮

The main result of the section is the following.

Theorem 3.5. *Let $G=(V, E)$ be a 4-connected bipartite graph and suppose $G'=(V', E)$ is a non-bipartite connected graph such that $F(G)=F(G')$. Then G' is obtained from G by either a rolling or an identification.*

Proof. If $|V|\leqq 4$, then the result is easily verified. Thus, assume $|V|\geqq 5$. Observe that the rank function definition implies $|V'|=|V|-1$.

Suppose that for every $u \in V$ the graph $G' \setminus \mathrm{st}_G(u)$ is non-bipartite, where $\mathrm{st}_G(u)$ denotes the set of edges of G incident to u. Choose $v \in V$, and define $H = G \setminus v$ and $H' = G' \setminus \mathrm{st}_G(v)$. Since G is 4-connected, H is 3-connected. Observe that $G \setminus \mathrm{st}_G(v)$ has exactly two bipartite components, H and the isolated vertex v. Thus, $\mathrm{st}_G(v)$ is a cocircuit of $F(G)$, and so H' has exactly one bipartite component, say B.

Since H is 3-connected, every pair of edges is in a cycle of H, and thus a circuit of $F(H)$. Therefore, every pair of edges of H' is in a circuit of $F(H')$. Suppose B has an edge, say e. Since H' is non-bipartite and B is bipartite, there exists an edge f of H' that is not in B. Evidently, $\{e, f\}$ is not contained in a circuit of $F(H')$, a contradiction. Thus, B is an isolated vertex, say v'. Moreover, $\mathrm{st}_{G'}(v') \subseteq \mathrm{st}_G(v)$. Since G' is connected, $\mathrm{st}_{G'}(v') \neq \emptyset$. Thus, $\mathrm{st}_{G'}(v')$ contains a cocircuit of $F(G')$ and since no cocircuit of a matroid properly contains another, $\mathrm{st}_{G'}(v') = \mathrm{st}_G(v)$. Since v was arbitrarily chosen, for every vertex $u \in V$, there exists a vertex $u' \in V'$ such that $\mathrm{st}_G(u) = \mathrm{st}_{G'}(u')$. This implies $|V| = |V'|$, a contradiction. Thus, without loss of generality, the graph H' can be assumed to be bipartite. Moreover, since H' has exactly one bipartite component, H' is connected.

Let u, u_1, u_2, and u_3 be distinct vertices of H such that u_1, u_2, and u_3 are adjacent to v in G. By (3.4), there exists a $(u, \{u_1, u_2, u_3\})$-fan F of H. Let P_i be the edge set of the (u, u_i)-path in $H[F]$, for $1 \leqq i \leqq 3$. Then $H[P_1]$, $H[P_2]$ and $H[P_3]$ pairwise have only the vertex u in common. Let e_i be an edge of G joining u_i and v, for $1 \leqq i \leqq 3$. Then $C_{ij} = P_i \cup P_j \cup \{e_i, e_j\}$ is a circuit of $F(G)$, for $1 \leqq i < j \leqq 3$.

Since H' is bipartite and H is 3-connected, by (3.3) $H = H'$. Thus, F is a $(u, \{u_1, u_2, u_3\})$-fan of H'. Since $|V'| = |V| - 1$, $V(H') = V'$.

One of the following is claimed to be true for G', $T = \{u_1, u_2, u_3\}$, $\{e_1, e_2, e_3\}$, and $t = 3$.

(i) e_i is a loop at u_i, for $1 \leqq i \leqq t$.
(ii) e_1 (say) is a loop at u_1, and e_i joins u_i and u_1, for $2 \leqq i \leqq t$, or
(iii) there exists a vertex $z \notin T$ such that e_i joins u_i and z, for $1 \leqq i \leqq t$.

The first step is to verify the following: if e_i is a loop of G', then e_i is a loop at u_i, for $1 \leqq i \leqq 3$. Suppose, for example, e_1 is a loop of G', and e_1 is not incident to u_1 or u_2, but rather to a vertex z. Let R be a (u_1, u_2)-path of H that avoids z. Then $R \cup \{e_1, e_2\}$ is a circuit of $F(G)$, but not of $F(G')$, a contradiction. Thus, e_1 is incident to, in G', u_1 or u_2. Likewise, e_1 is incident to, in G', u_1 or u_3. Thus, e_1 is a loop at u_1.

The next step is verify the following: if e_i is a loop of G' and e_j is not, then e_j joins u_i and u_j, for $1 \leqq i, j \leqq 3$. Suppose, for example, e_1 is a loop (at u_1, by the above) and e_2 joins z and z', with $z \neq z'$ and $\{z, z'\} \neq \{u_1, u_2\}$. Without loss of generality, assume $z \notin \{u_1, u_2\}$. Let R be a (u_1, u_2)-path of H that avoids z. Then $R \cup \{e_1, e_2\}$ is a circuit of $F(G)$, but not of $F(G')$, a contradiction.

The previous 2 cases show that if at least one of e_1, e_2 or e_3 is a loop, then either (i) or (ii) holds.

Now consider the case when none of e_1, e_2 or e_3 are loops of G'. Since C_{12} is a circuit of $F(G)$, e_1 is incident to exactly one of u_1 and u_2, and since C_{13} is a circuit of $F(G)$, e_1 is incident to exactly one of u_1 and u_3. Thus, e_i is incident to u_i, for $1 \leqq i \leqq 3$. Denote by z_i the other end of e_i, for $1 \leqq i \leqq 3$. If $z_1 \neq z_2$, then $R \cup \{e_1, e_2\}$ is a circuit of $F(G)$, but not $F(G')$, where R is a (u_1, u_2)-path of H avoiding z_1. Thus $z_1 = z_2 = z_3$, and so (iii) holds.

By induction on $|T|$, either property (i), (ii) or (iii) holds for $T=\{u_1, u_2, \dots, u_t\}$ and $\{e_1, \dots, e_t\}$, where $u_1, \dots, u_t$ are the vertices of G incident to v, and where e_i joins v and u_i, for $1 \leqq i \leqq t$. The theorem follows easily by obsevring that (i) implies that G' is obtained from G by a rolling, and (ii) and (iii) imply that G' is otbained from G by an identification. ▮

4. Concluding Remarks

Theorem (3.5) is not true if the 4-connectivity assumption is relaxed; the graphs in Figure 2 have the same factor matroid, but are not related by either a rolling or an identification. I have found several other factor-matroid-preserving operations and I conjecture that there is a finite list of such operations that yield all non-bipartite graphs having that same factor matroid as a given bipartite graph. The related problem of determining whether two non-bipartite graphs have the same factor matroid seems more difficult.

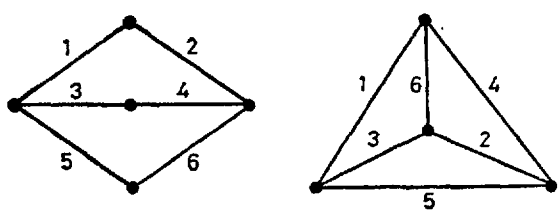

Fig. 2

Acknowledgement. I would like to thank both referees for their many helpful suggestions.

References

[1] J. A. BONDY and U. S. R. MURTY, *Graph Theory with Applications*, North-Holland, New York, 1976.
[2] M. DOOB, An interrelation between line graphs, eigenvalues and matroids, *Journal of Combinatorial Theory, Series B* **15** (1973), 40—50.
[3] J. M. S. SIMÕES-PEREIRA, On matroids on edge sets of graphs with connected subgraphs as circuits II, *Discrete Mathematics* **12** (1975), 55—78.
[4] W. T. TUTTE, On chain-groups and the factors of graph, *Algebraic Methods in Graph Theory, Colloquia Mathematica Societatis János Bolyai*, Szeged, Hungary, 1978.
[5] D. J. A. WELSH, *Matroid Theory*, Academic Press, London, 1976.
[6] H. WHITNEY, 2-Isomorphic graphs, *American Journal of Mathematics* **55** (1933), 245—254.
[7] T. ZASLAVSKY, Signed graphs, *Discrete Applied Mathematics* **4** (1982), 47—74.

Donald K. Wagner

School of Industrial Engineering
Purdue University
West Lafayette, Indiana 47907, U.S.A.